

# INTERPOLATION AND PEAK FUNCTIONS FOR THE NEVANLINNA AND SMIRNOV CLASSES

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**ABSTRACT.** It is known (implicit in [HMNT]) that when  $\Lambda$  is an interpolating sequence for the Nevanlinna or the Smirnov class then there exist functions  $f_\lambda$  in these spaces, with uniform control of their growth and attaining values 1 on  $\lambda$  and 0 in all other  $\lambda' \neq \lambda$ . We provide an example showing that, contrary to what happens in other algebras of holomorphic functions, the existence of such functions does not imply that  $\Lambda$  is an interpolating sequence.

## 1. INTRODUCTION

Consider the Nevanlinna class

$$N = \{f \in H(\mathbb{D}) : \lim_{r \rightarrow 1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi} < +\infty\},$$

which is a complete metric space with the distance defined by

$$d(f, g) = \lim_{r \rightarrow 1} \int_0^{2\pi} \log(1 + |f(re^{i\theta}) - g(re^{i\theta})|) \frac{d\theta}{2\pi}.$$

**Definition.** A sequence  $\Lambda \subset \mathbb{D}$  is a (free) interpolating sequence for  $N$  if the space of traces  $N|_\Lambda$  is ideal, that is, whenever  $f \in N$  and  $\{\omega_\lambda\}_{\lambda \in \Lambda}$  is a bounded sequence there exists  $g \in N$  such that  $g(\lambda) = \omega_\lambda f(\lambda)$ ,  $\lambda \in \Lambda$ . We shall write  $\Lambda \in \text{Int } N$ .

Since  $N$  is an algebra, it is easily seen that  $\Lambda \in \text{Int } N$  if and only if for every bounded sequence  $\{v_\lambda\}_{\lambda \in \Lambda}$  there exists  $f \in N$  such that  $f(\lambda) = v_\lambda$ ,  $\lambda \in \Lambda$ . In particular, if  $\Lambda \in \text{Int } N$  there exist functions  $f_\lambda \in N$  interpolating the values

$$\delta_{\lambda, \lambda'} = \begin{cases} 1 & \text{if } \lambda' = \lambda \\ 0 & \text{if } \lambda' \neq \lambda. \end{cases}$$

Moreover this can be achieved with functions  $f_\lambda$  such that  $\sup_{\lambda \in \Lambda} d(f_\lambda, 0) < \infty$ , as can be seen by going through the details of the proof of [HMNT, Theorem 1.2].

Note that for other algebras of holomorphic functions the analogous size control of these  $f_\lambda$  is an immediate consequence of the open mapping theorem applied to the restriction operator. This is the case for  $H^\infty$ , the algebra of bounded holomorphic functions, the Korenblum algebra  $A^{-\infty}$ , or the Smirnov class  $N^+$  (see below). However,  $N$  is not even a topological vector space [ShSh], so no open mapping theorem can be used here.

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Conversely, for  $H^\infty$  and  $A^{-\infty}$  (and for other Banach spaces), the existence of functions  $f_\lambda$  with uniform control of their size and interpolating the values  $\delta_{\lambda, \lambda'}$  implies that  $\Lambda$  is an interpolating sequence (see [Gar07, Chap.VII], [Ma, Lemma 2.3]). We provide an example showing that this is not the case for  $N$  nor  $N^+$ . For the Nevanlinna class, we have:

**Theorem 1.1.** *Let  $\Lambda$  be a sequence in  $\mathbb{D}$ . Then,*

- (a) *If  $\Lambda \in \text{Int } N$ , then there exist  $C > 0$  and functions  $f_\lambda \in N$  such that*
  - (i)  $d(f_\lambda, 0) \leq C$ ,
  - (ii)  $f_\lambda(\lambda') = \delta_{\lambda, \lambda'}$ ,  $\lambda' \in \Lambda$ .
- (b) *The converse fails: there is a sequence  $\Lambda \notin \text{Int } N$  for which there exist  $C > 0$  and  $f_\lambda \in N$  satisfying (i) and (ii).*

On the other hand, the Smirnov class  $N^+$  is defined by

$$N^+ = \left\{ f \in N : \lim_{r \rightarrow 1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi} = \int_0^{2\pi} \log^+ |f^*(e^{i\theta})| \frac{d\theta}{2\pi} \right\},$$

where  $f^*(e^{i\theta}) = \lim_{r \rightarrow 1} f(re^{i\theta})$  (which exists a.e.  $\theta \in [0, 2\pi)$ ).

Since  $N^+$  is an  $F$ -space ([Ya, Lemma 1]), an application of the open mapping theorem for such spaces (see [Ru73, 2.11, p.47]) shows that there exist  $C > 0$  and functions  $f_\lambda \in N^+$  satisfying (i) and (ii) from Theorem 1.1. But more can be said. Denote by  $\mathcal{F}$  the class of convex, increasing functions  $\psi : [0, +\infty) \rightarrow [0, +\infty)$  such that  $\lim_{t \rightarrow +\infty} \psi(t)/t = +\infty$ . It is known that if  $f \in N^+$ , there exists  $\psi \in \mathcal{F}$ , depending on  $f$ , such that

$$\int_0^{2\pi} \psi \left[ \log(1 + |f^*(e^{i\theta})|) \right] \frac{d\theta}{2\pi} < +\infty.$$

**Theorem 1.2.** *Let  $\Lambda$  be a sequence in  $\mathbb{D}$ . Then,*

- (a) *If  $\Lambda \in \text{Int } N^+$ , then there exist  $\psi \in \mathcal{F}$ ,  $C > 0$  and functions  $f_\lambda \in N^+$  such that*
  - (i)  $\int_0^{2\pi} \psi \left[ \log(1 + |f_\lambda^*(e^{i\theta})|) \right] \frac{d\theta}{2\pi} \leq C$ ,
  - (ii)  $f_\lambda(\lambda') = \delta_{\lambda, \lambda'}$ ,  $\lambda' \in \Lambda$ .
- (b) *The converse fails: there is a sequence  $\Lambda \notin \text{Int } N^+$  for which there exist  $\psi \in \mathcal{F}$ ,  $C > 0$  and  $f_\lambda \in N^+$  satisfying (i) and (ii).*

As in the Nevanlinna case, part (a) is implicit in the proof of [HMNT, Theorem 1.3], while part (b) will follow from an explicit example.

## 2. PRELIMINARIES. INTERPOLATION IN THE NEVANLINNA AND SMIRNOV CLASSES

A complete description of the interpolating sequences for  $N$  and  $N^+$ , including a characterisation of the traces, was given in [HMNT, Theorems 1.2 and 1.3]. In particular, we will make use of the following geometric characterisation.

Let  $b_\lambda(z) = \frac{z-\lambda}{1-\bar{\lambda}z}$  be a Blaschke factor and  $B_\lambda(z) := \prod_{\lambda' \neq \lambda} \left(-\frac{|\lambda'|}{\lambda'}\right) b_{\lambda'}(z)$  the Blaschke product with one factor omitted. Given a finite measure  $\mu$  in  $\mathbb{T}$ , let

$$P[\mu](z) = \int_0^{2\pi} \frac{1 - |z|^2}{|z - e^{i\theta}|^2} d\mu(\theta)$$

denote its Poisson transform.

**Theorem A.** [HMNT] *Let  $\Lambda \subset \mathbb{D}$ .*

(a)  $\Lambda \in \text{Int } N$  *if and only if there exists  $\mu$  positive measure with finite mass on  $\mathbb{T}$  such that*

$$|B_\lambda(\lambda)| \geq e^{-P[\mu](\lambda)}, \quad \lambda \in \Lambda.$$

(b)  $\Lambda \in \text{Int } N^+$  *if and only if there exists  $w \geq 0$ ,  $w \in L^1(\mathbb{T})$ , such that*

$$|B_\lambda(\lambda)| \geq e^{-P[w](\lambda)}, \quad \lambda \in \Lambda.$$

In classical terminology, when  $w$  is a positive function in  $L^1(\mathbb{T})$ , the harmonic function  $u = P[w]$  is called *quasi-bounded*. According to [ArGa, Theorem 1.3.9, p.10], for any such functions there exists  $\psi \in \mathcal{F}$  such that

$$(1) \quad \sup_{r < 1} \int_0^{2\pi} \psi[u(re^{i\theta})] \frac{d\theta}{2\pi} = \int_0^{2\pi} \psi[w(e^{i\theta})] \frac{d\theta}{2\pi} < +\infty.$$

### 3. PROOF. NECESSITY.

As said before, the necessity of conditions (i) and (ii) in Theorems 1.1 and 1.2 is implicit in [HMNT]. We briefly recall how this goes.

Assume  $\Lambda \in \text{Int } N$  and let  $\mu$  be the measure given by Theorem A (a). Consider the function

$$g(z) = \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta).$$

Since  $\text{Re } g(z) = P[\mu](z) \geq 0$  we see that  $g \in N^+$  and also  $e^g \in N$ . Letting  $H = (2 + g)^2$  we have  $H \in N^+$  and

$$|H(\lambda)| \geq (2 + \text{Re } g(\lambda))^2 \geq \left(2 + \log \frac{1}{|B_\lambda(\lambda)|}\right)^2 = \left(1 + \log \frac{e}{|B_\lambda(\lambda)|}\right)^2,$$

whence letting  $\phi(t) = (1 + t)^{-2}$  we obtain, for any  $\lambda' \in \Lambda$ ,

$$|\delta_{\lambda\lambda'}| \leq |B_\lambda(\lambda)| \phi\left(\log \frac{e}{|B_\lambda(\lambda)|}\right) e^{P[\mu](\lambda)} |H(\lambda)|.$$

**Theorem.** [Gar77, Theorem 4] *Let  $\phi : [0, \infty) \rightarrow [0, \infty)$  be a decreasing function such that  $\int_0^\infty \phi(t) dt < \infty$ . There exists  $C > 0$  such that if  $\{v_\lambda\}_\lambda$  is a sequence with*

$$|v_\lambda| \leq |B_\lambda(\lambda)| \phi\left(\log \frac{e}{|B_\lambda(\lambda)|}\right) \quad \lambda \in \Lambda,$$

*then there exists  $F \in H^\infty$  with  $F(\lambda) = v_\lambda$ ,  $\lambda \in \Lambda$ , and  $\|F\| \leq C \int_0^\infty \phi(t) dt$ .*

Applying Garnett's theorem to the sequence  $\left\{ \frac{\delta_{\lambda\lambda'}}{e^{P[\mu](\lambda)}|H(\lambda)|} \right\}_{\lambda \in \Lambda}$  we obtain a constant  $C(\phi)$  and functions  $F_\lambda \in H^\infty$  with  $\|F_\lambda\|_\infty \leq C(\phi)$  and

$$F_\lambda(\lambda') = \frac{\delta_{\lambda\lambda'}}{e^{P[\mu](\lambda)}|H(\lambda)|}.$$

Defining

$$f_\lambda(z) = F_\lambda(z)e^{g(z)}H(z)$$

we finally have (ii) and

$$\log^+ |f_\lambda(z)| \leq \log C(\phi) + P[\mu](z) + \log^+ |H(z)|,$$

which implies (i).

The same proof for the Smirnov case provides interpolating functions  $f_\lambda \in N^+$  with

$$\log^+ |f_\lambda(z)| \leq \log C(\phi) + P[w](z) + \log^+ |H(z)|,$$

where  $w$  is given by Theorem A(b). Since  $H \in N^+$ , the subharmonic function  $\log^+ |H(z)|$  has a quasi-bounded harmonic majorant, and (i) in Theorem 1.2 follows from (1).

#### 4. PROOF. LACK OF SUFFICIENCY.

In order to construct examples of non-interpolating sequences satisfying (i) and (ii) in Theorems 1.1 and 1.2, consider the dyadic intervals on  $\mathbb{T}$ :

$$I_{n,k} = \{e^{2\pi i\theta} : k2^{-n} \leq \theta < (k+1)2^{-n}\}, \quad n \in \mathbb{N}, \quad k = 0, \dots, 2^n - 1.$$

Let us prove first Theorem 1.1 (b).

Consider the sequence  $A$  defined in the following way: on the ray terminating at an end of a dyadic interval of the  $n$ -th generation (and which is not an end of an interval in a previous generation) consider the dyadic sequence with radii  $1 - 2^{-m}$ , starting at  $m = 2n$ . Explicitly

$$A = \{a_m^{n,k}\}_{\substack{n \in \mathbb{N} \\ 0 \leq k < 2^n - 1, k \text{ odd} \\ m \geq 2n}} \quad a_m^{n,k} = (1 - 2^{-m})e^{2\pi i k 2^{-n}}.$$

Now, to each  $a \in A$  associate a point  $b$  on the same ray and so that

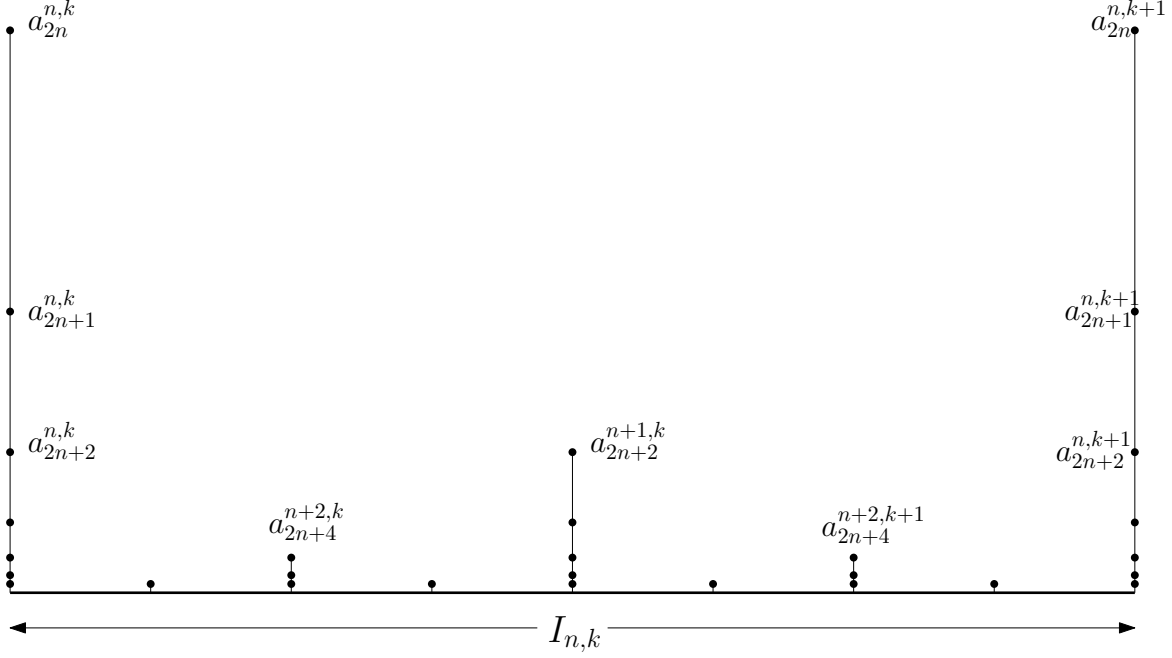
$$(2) \quad \varrho(a, b) := \left| \frac{a - b}{1 - \bar{a}b} \right| \simeq \exp \left( -\frac{1}{1 - |a|} \right).$$

We thus obtain a sequence  $B$ , which can be explicitly given by

$$B = \{b_m^{n,k}\}_{\substack{n \in \mathbb{N} \\ 0 \leq k < 2^n - 1, k \text{ odd} \\ m \geq 2n}} \quad b_m^{n,k} = (1 - e^{-2^m})a_m^{n,k}.$$

**Lemma 4.1.** *The sequences  $A$  and  $B$  are both interpolating for  $H^\infty$ .*

Recall that  $A \in \text{Int } H^\infty$  means that there exists  $C > 0$  (the interpolation constant) such that for every bounded sequence  $\{v_a\}_{a \in A}$  there is  $F \in H^\infty$  with  $\|F\|_\infty \leq C\|\{v_a\}\|_\infty$  and  $F(a) = v_a$ ,  $a \in A$ .

FIGURE 1. Representation of the points of  $A$  “above” the interval  $I_{n,k}$ 

*Proof.* Let us see that  $A \in \text{Int } H^\infty$ . By Carleson’s theorem ([Gar07, Theorem 1.1, Chap.VII]) it is enough to see that  $A$  is separated in the pseudo-hyperbolic metric  $\varrho$ , which is obvious from the definition, and that  $\nu = \sum_{a \in A} (1 - |a|) \delta_a$  is a Carleson measure.

Let us see first that  $A$  is a Blaschke sequence:

$$\sum_{a \in A} 1 - |a| = \sum_{n=1}^{\infty} \sum_{k=0}^{2^n-1} \sum_{m \geq 2n} 2^{-m} \simeq \sum_{n=1}^{\infty} 2^n 2^{-2n} < +\infty$$

In order to see that  $\nu$  is a Carleson measure we have to prove that there exists  $C > 0$  such that  $\nu(Q(I)) \leq C|I|$  for all  $I$  interval in  $\mathbb{T}$ , where  $Q(I) = \{re^{i\theta} : 1 - r < |I|, e^{i\theta} \in I\}$  is the associated Carleson box. It is enough to consider the case where  $I$  a dyadic interval. Thus let  $I = I_{n,k}$  and

$$Q(I_{n,k}) = \{re^{i\theta} : r > 1 - 2^{-n}, \theta \in 2\pi[k2^{-n}, (k+1)2^{-n}]\}.$$

By construction, in  $Q(I_{n,k})$  there are  $2^j$  rays of the  $(n+j)$ -th generation. Hence

$$\sum_{a \in A \cap Q(I_{n,k})} 1 - |a| \simeq \sum_{j=1}^{\infty} 2^j \sum_{m \geq 2(n+j)} 2^{-m} \simeq \sum_{j=1}^{\infty} 2^{-2n-j} \lesssim 2^{-n} = |I_{n,k}|.$$

□

Define  $\Lambda = A \cup B$ . Let us see first that there exist  $C > 0$  and  $f_\lambda$  satisfying (i) and (ii) in Theorem 1.1.

Fix  $\lambda = a_m^{n,k}$  and denote by  $\tilde{\lambda} = b_m^{n,k}$  its “twin”. As just seen, there exist  $C > 0$  and  $P_\lambda^A \in H^\infty$  such that

- $\|P_\lambda^A\|_\infty \leq C$ ,
- $P_\lambda^A(\lambda) = 1$  and  $P_\lambda^A(a_{m'}^{n',k'}) = 0 \quad \forall (n', k', m') \neq (n, k, m)$ .

As in the proof of Lemma 4.1, we can see that  $(B \setminus \{\tilde{\lambda}\}) \cup \{\lambda\}$  is also in  $\text{Int } H^\infty$ , and with interpolation constant  $C > 0$  independent of  $\lambda$ . Therefore there exist  $P_\lambda^B \in H^\infty$  such that

- $\|P_\lambda^B\|_\infty \leq C$ ,
- $P_\lambda^B(\lambda) = 1$  and  $P_\lambda^B(b_{m'}^{n',k'}) = 0 \quad \forall (n', k', m') \neq (n, k, m)$ .

Define finally

$$(3) \quad f_\lambda := c_\lambda P_\lambda^A P_\lambda^B b_{\tilde{\lambda}} e^{g_\lambda},$$

where

$$g_\lambda(z) = \frac{\lambda^* + z}{\lambda^* - z} \quad (\lambda^* = \lambda/|\lambda|)$$

and  $c_\lambda$  is chosen so that  $f_\lambda(\lambda) = 1$ .

Notice that, by construction, (ii) in Theorem 1.1 holds. In order to see (i) notice that (2) gives

$$|c_\lambda| = \frac{1}{|b_{\tilde{\lambda}}(\lambda)|} \exp\left(-\frac{1+|\lambda|}{1-|\lambda|}\right) \simeq \exp\left(\frac{1}{1-|\lambda|} - \frac{1+|\lambda|}{1-|\lambda|}\right) \lesssim 1.$$

Then

$$\log |f_\lambda(z)| = \log |c_\lambda| + \log |P_\lambda^A(z)| + \log |P_\lambda^B(z)| + \log |b_{\tilde{\lambda}}(z)| + \text{Re } g_\lambda(z)$$

and therefore

$$\log^+ |f_\lambda(z)| \leq \log \|c\|_\infty + 2 \log C + \text{Re } g_\lambda(z).$$

Since

$$\sup_{r < 1} \int_0^{2\pi} \text{Re } g_\lambda(re^{i\theta}) \frac{d\theta}{2\pi} = \sup_{r < 1} \int_0^{2\pi} \frac{1-r^2}{|re^{i\theta} - \lambda^*|^2} \frac{d\theta}{2\pi} = 1,$$

we have (i), as desired.

Let us see now that  $\Lambda \notin \text{Int } N$  by seeing that there is no  $\mu$  satisfying the condition of Theorem A(a). Since

$$\log \frac{1}{|B_\lambda(\lambda)|} \simeq \frac{1}{1-|\lambda|} \quad \lambda \in \Lambda,$$

such  $\mu$  should satisfy in particular (fixed any  $n, k$ )

$$1 \lesssim (1 - |a_m^{n,k}|) P[\mu](a_m^{n,k}), \quad \forall m \geq 2n.$$

This would force the measure  $\mu$  to satisfy  $1 \lesssim \mu\{e^{2\pi i k 2^{-n}}\}$  for all  $n, k$  ([ShSh, Theorem 2.2]), and therefore  $\mu(\mathbb{T})$  could not be finite.

Let us prove now Theorem 1.2 (b). In the same construction done for  $N$  consider a sequence made of the “first” couple of points of each ray, and with a slightly bigger separation. More precisely, let  $\tilde{\Lambda} = \tilde{A} \cup \tilde{B}$ , where

$$\tilde{A} = \{a_{n,k}\}_{\substack{n \in \mathbb{N} \\ 0 \leq k < 2^n - 1, \, k \text{ odd}}} \quad a_{n,k} = (1 - 2^{-2n}) e^{2\pi i k 2^{-n}}.$$

and  $\tilde{B} = \{b_{n,k}\}_{n,k}$  is so that  $a_{n,k}$  and  $b_{n,k}$  are on the same ray and

$$(4) \quad \varrho(a_{n,k}, b_{n,k}) = \exp \left( -\frac{1}{(1 - |a_{n,k}|) \log_2(\frac{1}{1 - |a_{n,k}|})} \right) = \frac{2^{2n}}{2n}.$$

As in Lemma 4.1,  $\tilde{A}$  and  $\tilde{B}$  are  $H^\infty$ -interpolating sequences, so there exist bounded peak functions  $P_{\tilde{A}}, P_{\tilde{B}}$  with the same properties as before.

Given  $\lambda \in \tilde{\Lambda}$ , let  $I_\lambda$  denote the Privalov “shadow” of  $\lambda$  on  $\mathbb{T}$ , that is

$$I_\lambda = \{e^{i\theta} : |e^{i\theta} - \lambda| \leq 2(1 - |\lambda|)\}.$$

Let

$$w_\lambda(\theta) = \frac{C_0}{(1 - |\lambda|) \log_2(\frac{1}{1 - |\lambda|})} \chi_{I_\lambda}(e^{i\theta}),$$

where  $c$  is a universal constant to be chosen later, and consider

$$g_\lambda(z) = \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} w_\lambda(\theta) \frac{d\theta}{2\pi}.$$

Notice that  $\|w_\lambda\|_{L^1(\mathbb{T})} \simeq c(\log_2(\frac{1}{1 - |\lambda|}))^{-1} \lesssim 1$  and

$$\operatorname{Re} g_\lambda(\lambda) = P[w_\lambda](\lambda) \simeq \frac{1}{1 - |\lambda|} \int_{I_\lambda} w_\lambda(\theta) d\theta \geq \frac{C_1 C_0}{(1 - |\lambda|) \log_2(\frac{1}{1 - |\lambda|})}.$$

Now define  $f_\lambda$  as in (3), with these new  $g_\lambda$ . Again, it is clear that (ii) in Theorem 1.2 holds. Also,  $\{c_\lambda\}_{\lambda \in \Lambda}$  is bounded if  $C_0$  is chosen appropriately: if  $\lambda = a_m^{n,k}$ ,

$$\log |c_\lambda| = \log \frac{1}{\varrho(a_{n,k}, b_{n,k})} - \operatorname{Re} g_{a_{n,k}}(a_{n,k}) \leq \frac{1 - C_1 C_0}{(1 - |a_{n,k}|) \log_2(\frac{1}{1 - |a_{n,k}|})} \leq 0.$$

In order to see (i) notice that, as before, there exists  $\tilde{C} > 0$  such that

$$\log^+ |f_\lambda| \leq \log \|c\|_\infty + 2 \log C + P[w_\lambda] \leq \tilde{C} + P[w_\lambda].$$

Therefore, by (1), for  $\lambda = a_m^{n,k}$  and taking  $\psi(t) = (1 + t) \log(1 + t) \in \mathcal{F}$  we get (i):

$$\begin{aligned} \int_0^{2\pi} \psi[\log^+ |f_\lambda^*(e^{i\theta})|] \frac{d\theta}{2\pi} &\leq \int_0^{2\pi} \psi[\tilde{C} + \frac{C_0 \chi_{I_\lambda}(\theta)}{(1 - |\lambda|) \log_2(\frac{1}{1 - |\lambda|})}] \frac{d\theta}{2\pi} \\ &= \int_{\theta \notin I_\lambda} \psi(\tilde{C}) \frac{d\theta}{2\pi} + \int_{I_\lambda} \psi[\tilde{C} + \frac{C_0}{(1 - |\lambda|) \log_2(\frac{1}{1 - |\lambda|})}] \frac{d\theta}{2\pi} \\ &\lesssim 1 + (1 - |\lambda|) \psi[\frac{2C_0}{(1 - |\lambda|) \log_2(\frac{1}{1 - |\lambda|})}] \\ &\lesssim 1 + \frac{1}{\log_2(\frac{1}{1 - |\lambda|})} \log_2 \left( \frac{2C_0}{(1 - |\lambda|) \log_2(\frac{1}{1 - |\lambda|})} \right) \lesssim 1. \end{aligned}$$

Let us finish by proving that  $\Lambda \notin \text{Int } N^+$ . Assume that there is  $w \in L^1(\mathbb{T})$  satisfying Theorem A (b). Then

$$\begin{aligned} \frac{1}{(1 - |a_{n,k}|) \log_2(\frac{1}{1 - |a_{n,k}|})} &= \log \frac{1}{\varrho(a_{n,k}, b_{n,k})} \leq \log \frac{1}{|B_{a_{n,k}}(a_{n,k})|} \\ &\leq P[w](a_{n,k}) = \int_0^{2\pi} \frac{1 - |a_{n,k}|^2}{|a_{n,k} - e^{i\theta}|^2} w(\theta) \frac{d\theta}{2\pi}, \end{aligned}$$

and therefore

$$\sum_{n \geq 1} \sum_{k=0}^{2^n-1} \frac{1}{2^n} \simeq \sum_{n \geq 1} \sum_{k=0}^{2^n-1} \frac{1}{\log_2(\frac{1}{1 - |a_{n,k}|})} \lesssim \int_0^{2\pi} \sum_{n,k} \frac{(1 - |a_{n,k}|^2)^2}{|a_{n,k} - e^{i\theta}|^2} w(\theta) \frac{d\theta}{2\pi}.$$

We will have a contradiction as soon as we prove that

$$\sup_{\theta \in [0, 2\pi)} \sum_{n,k} \frac{(1 - |a_{n,k}|^2)^2}{|a_{n,k} - e^{i\theta}|^2} < \infty.$$

With no loss of generality assume that  $e^{i\theta} = 1$  and that the dyadic intervals  $I_{n,k}$ ,  $0 \leq k < 2^n - 1$ , are ordered so that  $e^{i\theta} \in I_{n,0}$ . Then we have

$$\begin{aligned} |e^{i\theta} - a_{n,k}|^2 &= |1 - a_{n,k}|^2 \simeq (1 - |a_{n,k}|)^2 + |e^{2\pi i k 2^{-n}} - 1|^2 \\ &\simeq (2^{-2n})^2 + (k 2^{-n})^2 = 2^{-2n}(2^{-2n} + k^2). \end{aligned}$$

Since for each  $e^{i\theta}$  there is only one  $a_{n,k}$  with  $1 - |a_{n,k}| \simeq |e^{i\theta} - a_{n,k}|$ , we have then

$$\sum_{n,k} \frac{(1 - |a_{n,k}|^2)^2}{|a_{n,k} - e^{i\theta}|^2} \simeq \sum_{n \geq 1} \sum_{k=1}^{2^n-1} \frac{2^{-4n}}{2^{-2n} k^2} \simeq \sum_{n \geq 1} 2^{-2n}.$$

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